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position, modulation theory, eigenvalue dynamics, stability analysis, and shock wave theory. Models analyzed include the integrable as well as the generalized nonintegrable Korteweg-de Vries equation, the (modulationally unstable) nonlinear Schroedinger equation, and particle chains under various types of forcing.

Semiconductor instabilities are also investigated, which lead to the generation of time-periodic waves in semiconductors upon appropriate dc voltage bias. The nature of the instability that drives such time periodic behavior is explained and the phenomena are analyzed and understood by the use of analytical and computational means.

**THE GENERATION AND PROPAGATION OF OSCILLATIONS  
IN NONLINEAR SYSTEMS**

**FINAL PROGRESS REPORT**

**ARTHOR**  
STEPHANOS VENAKIDES

**DATE**  
September 19, 1996

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Stephanos Venakides

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The propagation of dispersive nonlinear waves in continuous and discrete media is investigated. The small scale dispersive oscillations are averaged out and a complete set of modulation equations that describe the evolution of the macroscopic quantities are derived and in special cases solved. At the level of greater refinement, detailed information on the small scale structure is obtained in integrable models. This is made possible by the development of a powerful new technique that leads to the explicit asymptotic solution of Riemann-Hilbert problems. Other techniques employed include, Liapounov-Schmidt decomposition, modulation theory, eigenvalue dynamics, stability analysis, and shock wave theory. Models analyzed include the integrable as well as the generalized nonintegrable Korteweg-de Vries equation, the (modulationally unstable) nonlinear Schroedinger equation, and particle chains under various types of forcing.

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# THE GENERATION AND PROPAGATION OF OSCILLATIONS IN NONLINEAR SYSTEMS

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September 17, 1996

## 1 Statement of the Problem

### 1.1 Dispersive Waves

Nonlinear dispersive wave motion manifests itself in many physical systems such as water waves and solitonic fiber-optical transmission. It has application in the electron flow analysis in semiconductors due to the dispersive character of the quantum hydrodynamic model; it is also relevant in superconductivity and superfluidity where one obtains Ginsburg-Landau equations of dispersive character. Finally, dispersive vibrations occur naturally in discrete media such as molecular lattices.

The focus of the present research has been to analyze the propagation of dispersive waves in both continuous and discrete media using the simplest model systems in which nonlinearity and dispersion coexist. In the dispersive systems studied, there is interaction between two different space-time scales, the small scale consisting of strongly nonlinear single or multiphase travelling waves whose wave parameters vary in the large scale. Such a situation arises generically in the solution of many nonlinear dispersive pde and ode systems. As a rule, the presence of a small dispersive term smoothens the shocks produced by the nonlinearity through the generation of rapid oscillations, that travel and spread over large space regions. Thus, an oscillatory microstructure may typically arise out of non-oscillatory initial data.

The first objective in the PI's approach is to average out the small scale dispersive oscillations and derive and solve a closed system of effective equations for macroscopic quantities. At the level of greater refinement, the goal is to obtain detailed information on the small scale structure e.g. resolve the local wave structure. Integrable and nonintegrable models have been used. In the integrable case, the powerful techniques of inverse scattering are further developed to obtain very detailed information on the behavior of the models. In the nonintegrable models, a combination of asymptotic techniques and rigorous iterative methods are employed. In both cases the emphasis has been on explaining and calculating the sharp phenomena that are observed in numerical experiments.

## 1.2 Waves in Semiconductors

Semiconductor instabilities are investigated, which lead to the generation of time-periodic waves in semiconductors upon appropriate dc voltage bias. An example of such a time-periodic phenomenon is the Gunn effect, in which a solitary pulse, born at the one end of a Gallium Arsenide (GaAs) sample under appropriate dc voltage bias, travels through the sample to vanish into the other end just as a new pulse that repeats the phenomenon is born at the first end. The goal is to study the nature of the instability that triggers such time periodic behavior and to use analytical and computational means to understand and analyze the phenomena.

## 2 Progress on Various Fronts

### 2.1 Continuous Media

#### 2.1.1 Integrable Small Dispersion Problems

The PI has developed, in collaboration with Deift and Zhou [DZVZ1, DVZ2], a rigorous asymptotic method for calculating the small scale oscillations, including phase shifts, WITHOUT apriori postulating them, i.e. without assuming that oscillations arise. The calculation is in the context of the initial value problem for the Korteweg-de Vries (KdV) equation  $u_t - 6uu_x + \epsilon^2 u_{xxx} = 0$  for small  $\epsilon$  and can be generalized to many other modulationally stable integrable systems. The new technique, is an extension of the steepest descent method for oscillatory Riemann-Hilbert Problems (RHP) developed by Deift and Zhou [DZ], in which the contour of the RHP is deformed, much in the spirit of contour deformation in the steepest descent method for the evaluation of integrals, and a contour of main contribution is identified. The extension consists in developing a systematic method for identifying the contour of main contribution. The new approach, while reproducing earlier results of Lax and Levermore [LL1, LL2], and of the PI [V1, V2, V3, V4], goes far beyond what was earlier known in resolving the small scale structure. In the course of identifying the contour of main contribution in the asymptotic analysis of the solution of the vector Riemann-Hilbert formulation of the inverse scattering problem, it introduces the nonlinear analogue of the eikonal equation that turns out to be a scalar Riemann-Hilbert problem. The final expression for the waveform, although in explicit form, contains a set of parameters whose calculation requires the solution of a complicated nonlinear system of algebraic (as opposed to differential) equations that contain hyperelliptic integrals and are subject to algebraic inequality constraints. In previous theories [LL2, V1, V4], this information was obtained through the solution of the modulation system [W1, W2, FFM, T], a system of nonlinear hyperbolic equations. Our algebraic system of equations and inequalities is essentially a complete set of first integrals for the modulation equations. We have proved the unique solvability of this system in a particular case.

#### 2.1.2 Nonintegrable Long -Time Problems

In collaboration with Kasturiarachi, the small dispersion limit for the generalized KdV equation  $u_t + u^3 u_x + u_{xxx} = 0$ , that has the KdV dispersive term, but a different nonlinear transport

term has been addressed. The emergence of oscillations with wave parameters varying slowly in space-time (modulational ansatz) is postulated and modulation equations that describe the evolution of the wave parameters in the large scale have been derived. The solution of the modulation system has been studied both numerically and analytically for Riemann initial data. The corresponding problem for integrable KdV was solved by Gurevich and Pitaevski [GP].

### 2.1.3 Weak Turbulence and the Focusing Nonlinear Schroedinger Equation

There are nonlinear dispersive systems in which the wave parameters of the small scale oscillations do not vary slowly in time but, due to a modulational instability, seem to evolve chaotically [BM]. Such behavior is displayed by both integrable and nonintegrable models, a prime model being the focusing Nonlinear Schroedinger Equation (NLS). The time-chaotic behavior triggered by the modulational instability that seems to feature a gas of solitons is often referred to as weak turbulence [ZS, LLV]. In current work in collaboration with Deift and Zhou it is found that the determination of the scattering data involves quantities that are small beyond all asymptotic orders. This is exemplified in our calculation [unpublished] on a simple model. The present focus is the study of the problem with shock initial data, that allows the explicit calculation of the scattering data, while it preserves the richness of the asymptotically chaotic behavior.

## 2.2 Discrete Media

### 2.2.1 Forced Discrete Dispersive Systems

In collaboration with Deift and Kriechrbauer, research has continued on the dispersive phenomena, that occur in forced nonlinear particle chains with nearest neighbor interactions, in particular on the generation of dispersive waves that transport energy away from a shock. Such phenomena were first observed by von Neuman [vN, see also HD] in his discretization of one dimensional gas dynamics. In subsequent famous experiments by Fermi, Pasta and Ulam [FPU], asymptotic recurrence was observed, contrary to the expectation that energy would be equipartitioned among the Fourier modes through nonlinear mode interaction. These experiments led to the discovery of the integrable chain [F]. In numerical experiments for various force laws Holian and Straub [HS] first observed critical behavior of the semi-infinite chain forced by a constant speed on the leading particle that tends to compress the chain. They observed that the chain tends to become asymptotically quiescent for low driving speeds, while it tends asymptotically to a binary oscillation when the driver speed exceeds a critical value. The shock problem, was subsequently studied analytically and numerically by Holian, Flaschka and McLaughlin [HFM]. In previous work the PI in collaboration with Deift and Oba [VDO] gave a complete explanation to the critical behavior in the case of an integrable chain. In the present research [DKV], a nonlinear theory is derived for a semi-infinite particle chain forced by a small amplitude time-PERIODIC driver at the leading particle. A frequency penetration threshold is found. Progressing single phase waves (integrable or nonintegrable chain) and multiphase waves (integrable chain) that match to the driver through a boundary layer are rigorously constructed. In the case of the strongly

nonlinear Toda chain, the integrable structure is perturbed by the existence of a driver and the integrability of the problem is still in question. The evolution of the conventional Lax operator [L] is no longer isospectral, and its spectrum has a nontrivial evolution in time. The evolution system of the spectrum is explicitly calculated, and an asymptotic theory is introduced that explains and predicts with great precision eigenvalue transport phenomena that are observed numerically.

### 2.2.2 The Shock Problem for Nonintegrable Chains

With student Filip, the critical behavior of the chain, forced by a constant velocity at the leading particle, is studied. A three parameter family of strongly nonlinear periodic traveling waves is rigorously constructed and a complete set of modulation equations for these waves is derived. Modulation theory is used to explain the phase transition discovered by Holian and Straub. In the course of this explanation, the modulation equations are solved numerically.

Modulation theory is used analytically to derive the behavior of the solution in the case of a rarefaction, in which the driver velocity is in the direction that tends to decompress the chain. The calculation leads to analyzing a p-system, and this analysis explains the following phase transition. Assume that the chain is doubly infinite (i.e. particle index  $n$  ranges over  $-\infty < n < \infty$ ), initial particle position is  $x_n(0) = dn$  where  $d$  is constant, and initial velocity is  $v_n = an$ ,  $a > 0$ . Then, for  $a < a_{crit}$ , the chain asymptotically has spacing  $d' > d$ , while for  $a > a_{crit}$  the chain breaks into different pieces with an ever increasing distance between them (cavitation). This phase transition has been previously explicitly calculated [DKKZ] in the case of an integrable chain but its explanation had remained open in the more general nonintegrable case.

### 2.2.3 Renormalization Phenomena in the Toda Chain

In collaboration with McDonald [McV], the well known explicit formula for the  $\tau$ -function of the finite Toda chain is made suitable for the solution of scattering problems in the infinite chain by a renormalization procedure. The  $\tau$ -function has essentially the form of a partition function, in the sense of statistical mechanics. As the number of particles tends to infinity the exponents in the expression also tend to infinity. A factor is extracted that blows up in the limit of infinitely many particles. The remaining expression still has the form of a partition function in which, however, the states are finite energy perturbations of the asymptotically infinite energy state that has been filtered out. The calculation is performed in the special context of the shock problem (it improves the [VDO] calculation by not neglecting the reflection coefficient a priori) but it also applies to all cases of scattering data.

## 2.3 Time Periodic Waves in Semiconductors

### 2.3.1 The Gunn Effect

In collaboration with Bonilla and Higuera [BHV] a linear stability analysis is performed of the stationary solution of a one-dimensional drift-diffusion model, used to describe the Gunn effect in Gallium Arsenide (GaAs). The Gunn effect is the time-periodic phenomenon

in which a solitary pulse, born at the one end of a GaAs sample under appropriate dc voltage bias, travels through the sample to vanish into the other end just as a new pulse is born that repeats the phenomenon. It is shown that for long semiconductor samples under dc voltage bias conditions and small diffusivity, the steady state may lose stability via a Hopf bifurcation. It is found that, in the limit of infinitely long samples, a quasicontinuum of oscillatory modes of the equation, linearized about the steady state, acquire a positive real part for voltages that are larger than a certain critical value, giving rise to the instability that constitutes the Gunn effect. The phenomenon is crucially dependent on the shape of the electron velocity vs electric field curve and in particular on the fact that over a range of electric field values the slope of the curve is negative, thus providing a negative differential resistance that drives the instability. The linear stability of the solitary pulse is established for an idealized electron velocity curve and zero diffusion.

### 2.3.2 Traveling Fronts in Semiconductor Superlattices

In collaboration with Bonilla, Kindelan and Moscoso, the generation and propagation of time periodic travelling fronts that produce self-sustained current oscillations observed experimentally in voltage biased semiconductor superlattices (SL) are analyzed by the use of asymptotic methods. The superlattice, a structure of alternating layers of two semiconductor materials, that corresponds to a potential consisting of a number of quantum wells, is examined in the limit of a large number of wells. The model, the continuum limit of a discrete model introduced by Bonilla, consists of a nonlinear hyperbolic equation for the electric field that reflects charge transport, coupled to an integral relation that imposes constant voltage bias. The equation is supplemented with appropriate shock and entropy conditions. For appropriate parameter values, a time-periodic solution is found in numerical simulations. Furthermore, an asymptotic analysis reveals traveling internal layers that are formed about shock waves and serve as moving boundaries of spatial domains in which the field depends on time but, up to exponentially small error, is uniform in space.

## 3 Publications Resulting from Research

1. (with L. L. Bonilla and F.J. Higuera), *The Gunn Effect: Instability of the Steady State and Stability of the Solitary Wave in Long Extrinsic Semiconductors*, SIAM Jrl. of Appl. Math. **54**, No. 6, (1994), 1521-1541.
2. (with P. Deift and X. Zhou), *The Collisionless Shock Region for the Long-Time Behavior of Solutions of the KdV Equation*, Comm. Pure Appl. Math. **47**, (1994), 199-206.
3. (with P. Deift and T. Kriecherbauer), *Forced Lattice Vibrations* Video-text:1995 MSRI preprint 003-95, paper: parts I, II, Comm. Pure Appl. Math., **48**, (1995), pp 1187-1250, 1251-1298.
4. (with L. Bonilla, M. Kindelan, M. Moscoso), *Periodic generation and Propagation of Travelling Fronts in dc Voltage Biased Semiconductor Superlattices*, SIAM Jrnl. Appl. Math., to appear.

5. (with P. Deift and X. Zhou), *New Results in the Small-Dispersion KdV by an Extension of the Steepest Descent Method for Riemann-Hilbert Problems Part I*, submitted, Proc. Natl. Ac. Sc.
6. (with M. McDonald), *Renormalization of the Toda  $\tau$  function and the Toda Shock Problem*, preprint.
7. (with B. Kasturiarachi), *The Small Dispersion Limit of the Nonintegrable, Generalized KdV Equation, with Riemann Initial Data*, in preparation.
8. (with A. Filip), *The Shock and rarefaction Problem for Nonintegrable Particle Chains*, in preparation.
9. (with P. Cheng and X. Zhou), *The Long Time Limit of the Sine-Gordon Equation*, in preparation.

## 4 Participating Scientific Personnel

Po-Jen Cheng, Graduate Student, Mathematics, Duke University. Anne-Marie Filip, graduate Student, Mathematics, Duke University.

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